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Symbolic-computation construction of transformations for a more generalized nonlinear Schrödinger equation with applications in inhomogeneous plasmas, optical fibers, viscous fluids and Bose-Einstein condensates

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Abstract. Currently, the variable-coefficient nonlinear Schrödinger (NLS)-typed models have attracted considerable attention in such fields as plasma physics, nonlinear optics, arterial mechanics and Bose-Einstein condensates. Motivated by the recent work of Tian et al. [Eur. Phys. J. B **47**, 329 (2005)], this paper is devoted to finding all the cases for a more generalized NLS equation with time- and space-dependent coefficients to be mapped onto the standard one. With the computerized symbolic computation, three transformations and relevant constraint conditions on the coefficient functions are obtained, which turn out to be more general than those previously published in the literature. Via these transformations, the Lax pairs are also derived under the corresponding conditions. For physical applications, our transformations provide the feasibility for more currently-important inhomogeneous NLS models to be transformed into the homogeneous one. Applications of those transformations to several example models are illustrated and some soliton-like solutions are also graphically discussed.

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1 Introduction

As one of the most important and "universal" nonlinear models of modern science, the standard nonlinear Schrödinger (NLS) equation [1,2],

$$i\,u_t + u_{xx} \pm 2\,|u|^2\,u = 0, \tag{1}$$

appears in many branches of physics and applied mathematics, including nonlinear quantum field theory, condensed matter and plasma physics, nonlinear optics and quantum electronics, fluid mechanics, theory of turbulence and phase transitions, biophysics, and star formation [2,3]. But in various real physical backgrounds, the variable-coefficient NLS-typed models are often considered to be more realistic than the standard one in that the variable coefficients can reflect the inhomogeneities of media and nonuniformities of boundaries [4–24]. Therefore, a lot of inhomogeneous NLS models with time- and/or space-dependent coefficients have been derived to describe a large variety of complicated situations in different physical contexts. Some physically-interesting examples are as follows:

(1) Space and laboratory plasmas are of current importance [25]. In a weakly inhomogeneous plasma, the propagation of envelope solitons obeys the following modified NLS equation [4–6],

$$i\,q_t + q_{xx} + \left[2\,|q|^2 - F(x,t)\right] q = 0, \tag{2}$$

where $F(x,t)$ denotes the inhomogeneity effect. When $F(x,t) = x$, equation (2) can be used to describe the low-frequency plasma dynamics in the case of resonant absorption of electromagnetic waves in fully ionized inhomogeneous plasmas [7] and soliton excitation by an incident electromagnetic wave in an inhomogeneous overdense plasma [8].

(2) In a real optical-fiber transmission system, the varying dispersion, Kerr nonlinearity, loss/gain and phase modulation are of practical importance with the consideration of the inhomogeneities resulting from such

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factors as the variation in the lattice parameters of the fiber media and fluctuation of the fiber's diameters [9]. Currently, many efforts have been devoted to the following two newly-proposed inhomogeneous NLS models [10,11]:

$$i\,u_z + \frac{D(z)}{2}\,u_{tt} + R(z)\,|u|^2\,u + i\,\Gamma(z)\,u + M(z)\,t^2\,u = 0, \tag{3}$$

and

$$i\,u_z + \frac{D(z)}{2}\,u_{tt} + R(z)\,|u|^2\,u + i\,\Gamma(z)\,u + i\,\Pi(z)\,|u|^2\,u = 0, \tag{4}$$

where $u(z,t)$ is the complex envelope of the electrical field in a comoving frame, z is the propagation distance, t is the retarded time, $D(z)$, $R(z)$, $\Gamma(z)$, $M(z)$ and $\Pi(z)$ are the inhomogeneous functions respectively related to the group velocity dispersion, self-phase modulation, linear loss/gain, phase modulation and nonlinear loss/gain. In practical applications, equations (3) and (4) and their various special forms [12–21] are of considerable value not only for the description of amplification/absorption and compression/broadening of optical solitons in inhomogeneous optical-fiber systems [10–16], but also for the study of stable transmission of managed solitons [11,18–21].

(3) In arterial mechanics [22], treating the arteries as thin-walled, linearly tapered, prestressed elastic tubes and blood as an incompressible viscous fluid, the governing equation which models the weakly nonlinear waves in such a fluid-filled elastic tube with variable radiuses is the dissipative NLS equation with variable coefficients [23],

$$\begin{aligned} &i\,U_\tau + \mu_1\,U_{\xi\xi} + \mu_2\,|U|^2\,U + i\,\mu_3\,\Theta\,\tau\,U_\xi \\ &\quad + \left(\mu_4\,\Theta^2\,\tau^2 + \mu_5\,\Theta\,\xi - \mu_6\,\Theta^2\,\tau\,\xi + i\,\mu_7\right)U = 0, \end{aligned} \tag{5}$$

where τ and ξ are the stretched coordinates from the time and axial coordinates after static deformation, U corresponds to the dynamical radial displacement upon such initial static deformation, Θ represents the tapering angle, μ_1 and μ_2 are the arterial-system parameters, μ_3, μ_4, μ_5 and μ_6 stand for the contribution of variable radiuses, whereas μ_7 gives the contribution of dissipation resulting from the viscosity of the fluid. Equivalently, equation (5) can be written as

$$i\,u_t + \mu_1\,u_{xx} + \mu_2\,|u|^2\,u + i\,\mu_7\,u - V(x,t)\,u = 0, \tag{6}$$

with the transformation

$$x = \xi - \frac{\mu_3\,\Theta\,\tau^2}{2}, \quad t = \tau, \quad u(x,t) = U(\xi,\tau), \tag{7}$$

where

$$\begin{aligned} V(x,t) &= \mu_6\,\Theta^2\,x\,t - \mu_5\,\Theta x \\ &\quad + \frac{\mu_3\,\mu_6\,\Theta^3\,t^3 - (2\,\mu_4 + \mu_3\,\mu_5)\,\Theta^2\,t^2}{2}. \end{aligned} \tag{8}$$

(4) At low temperatures, the dynamics of a repulsive quasi-one-dimensional Bose-Einstein condensate, oriented along the x axis, can be described by the one-dimensional time-dependent Gross-Pitaevskii equation [24],

$$i\,\hbar\,\psi_t = -\frac{\hbar^2}{2\,m}\,\psi_{xx} + \bar{V}(x)\,\psi + g(t)\,|\psi|^2\,\psi, \tag{9}$$

where ψ is the mean-field Bose-Einstein condensate wave function, m is the atomic mass, $g(t)$ is the nonlinearity coefficient accounting for the interatomic interaction, and $\bar{V}(x)$ represents the external potential assumed to be the usual harmonic trap.

In recent years, there has been a growing interest in studying the variable-coefficient nonlinear evolution equations (NLEEs) which provide a large family of powerful models for describing the real-world situations in many fields of physical and engineering sciences [25,26]. However, those time- and/or space-dependent coefficient functions often bring about an unexpected increase of investigation difficulties owing to the involvement of a great amount of integral and differential calculations which are unmanageable manually. Fortunately, with the development of computer sciences and technologies, symbolic computation as a new branch of artificial intelligence is becoming an important tool to analytically investigate the NLEEs and relevant solitonic phenomena [25–27]. The proposition of various methods based on the computerized symbolic computation [27] has made it exercisable to deal with the coefficient functions in the NLEEs under investigation.

It deserves to be specially noted that a straightforward and efficient method [12,28–32] for investigating the variable-coefficient NLEEs is to transform them into some known constant-coefficient equations with symbolic computation. For example, reference [12] has successfully transformed the following generalized variable-coefficient NLS equation,

$$i\,u_t + k(t)\,u_{xx} + l(t)\,|u|^2\,u = -i\,\Gamma(t)\,u, \tag{10}$$

which arises from space/laboratory plasmas, fluid dynamics and optical fibers, into the standard and cylindrical NLS equations under certain constraint conditions on the coefficient functions. In order to transform more currently-important inhomogeneous NLS models from various branches of physics into the homogeneous one, we in this paper consider a more generalized NLS equation with time- and space-dependent coefficients as below:

$$\begin{aligned} &i\,u_t + f(t)\,u_{xx} + [\,i\,g_1(t) + g_2(t)\,]\,|u|^2\,u \\ &\qquad + [\,i\,h_1(t) + h_2(x,t)\,]\,u = 0, \end{aligned} \tag{11}$$

where u is a complex function of x and t, $f(t)$, $g_1(t)$, $g_2(t)$ and $h_1(t)$ are all real analytical functions of t, and $h_2(x,t)$ is a real analytical function of x and t. Note that the above-mentioned six inhomogeneous NLS models (i.e., Eqs. (2–4), (6), (9) and (10)) are all the special cases of equation (11).

The structure of the present paper is as follows. In Section 2, with the aid of symbolic computation, we will (a) construct the transformations from equation (11) to the standard NLS equation, (b) find the relevant constraint conditions on the coefficient functions, and (c) compare our results with those previously reported. In Section 3, via the obtained transformations, we will derive the Lax pairs for equation (11) with the corresponding constraint conditions. At last, Section 4 will present the discussions about the applications of our transformations in some physical fields.

2 More general transformations to the standard NLS equation

Analogous to reference [12], we assume the transformation for equation (11) to be of the following general form

$$u = \alpha(t)\, e^{i\,\beta(x,t)}\, U\,[\, X(x,t),\, T(t)\,]\,, \tag{12}$$

where $\alpha(t) \neq 0$ and $\beta(x,t)$ are two real functions, while $X(x,t)$ with $X_x \neq 0$ and $T(t)$ are two complex functions which is different from the assumption of real functions in reference [12] (see format (11) there), because of the existence of the term $i\, g_1(t)\,|u|^2\, u$ in equation (11).

2.1 Symbolic manipulations

Directly inserting format (12) into equation (11) and requiring that $U(X,T)$ should satisfy the standard NLS equation, i.e.,

$$i\, U_T + U_{XX} \pm 2\,|U|^2\, U = 0, \tag{13}$$

one can get the following system of equations:

$$2\, f\, X_x^2 = \pm\, \alpha^2\, (i\, g_1 + g_2)\,, \tag{14}$$
$$2\, T' = \pm\, \alpha^2\, (i\, g_1 + g_2)\,, \tag{15}$$
$$i\, X_t + 2\, i\, f\, X_x\, \beta_x + f\, X_{xx} = 0, \tag{16}$$
$$\alpha\, h_1 + \alpha' + \alpha\, f\, \beta_{xx} = 0, \tag{17}$$
$$h_2 - \beta_t - f\, \beta_x^2 = 0, \tag{18}$$

where the prime sign denotes the differentiation with respect to t. Starting from equations (14–18), our intention in what follows is twofold: the first is to perform symbolic computation and work out the analytical expressions of $\alpha(t)$, $\beta(x,t)$, $X(x,t)$ and $T(t)$; the second is to find the constraint conditions on $f(t)$, $g_1(t)$, $g_2(t)$, $h_1(t)$ and $h_2(x,t)$.

By equations (14) and (15), we can firstly get the following results:

$$X = i\, \zeta_1(t)\, x + \zeta_2(t)\, x + i\, \eta_1(t) + \eta_2(t), \tag{19}$$
$$T = \pm\, \frac{i}{2} \int \alpha^2\, g_1\, dt \pm \frac{1}{2} \int \alpha^2\, g_2\, dt + \delta, \tag{20}$$

with the constraint conditions,

$$g_1 = \pm\, \frac{4\, f\, \zeta_1(t)\, \zeta_2(t)}{\alpha^2}, \tag{21}$$
$$g_2 = \pm\, \frac{2\, f\, \zeta_2^2(t) - 2\, f\, \zeta_1^2(t)}{\alpha^2}, \tag{22}$$

where δ is an arbitrary complex constant, $\zeta_1(t)$, $\zeta_2(t)$, $\eta_1(t)$ and $\eta_2(t)$ are four real functions to be further determined. Then, substituting equation (19) into equation (16), after calculation, the resulting equation can be equivalently split into

$$x\, \zeta_1' + \eta_1' + 2\, f\, \zeta_1\, \beta_x = 0, \tag{23}$$
$$x\, \zeta_2' + \eta_2' + 2\, f\, \zeta_2\, \beta_x = 0, \tag{24}$$

which are combined with equation (18), determining that $\beta(x,t)$ and $h_2(x,t)$ are both the general quadratic polynomials with respect to x of the forms

$$\beta = \beta_2(t)\, x^2 + \beta_1(t)\, x + \beta_0(t), \tag{25}$$
$$h_2 = \vartheta(t)\, x^2 + \theta(t)\, x + \sigma(t), \tag{26}$$

where $\vartheta(t)$, $\theta(t)$ and $\sigma(t)$ are all arbitrary real analytical functions of t, while $\beta_0(t)$, $\beta_1(t)$ and $\beta_2(t)$ are three functions to be further determined. Thus, equations (17), (18), (23) and (24) with the substitution of expressions (25) and (26) turn out to be

$$\alpha' + \alpha\, h_1 + 2\, \alpha\, f\, \beta_2 = 0, \tag{27}$$
$$\left(\vartheta - 4\, f\, \beta_2^2 - \beta_2'\right)\, x^2 + \left(\theta - 4\, f\, \beta_1\, \beta_2 - \beta_1'\right)\, x + \sigma - f\, \beta_1^2 - \beta_0' = 0, \tag{28}$$
$$\left(\zeta_1' + 4\, f\, \zeta_1\, \beta_2\right)\, x + \eta_1' + 2\, f\, \zeta_1\, \beta_1 = 0, \tag{29}$$
$$\left(\zeta_2' + 4\, f\, \zeta_2\, \beta_2\right)\, x + \eta_2' + 2\, f\, \zeta_2\, \beta_1 = 0, \tag{30}$$

which are solved for $\zeta_1(t)$, $\zeta_2(t)$, $\eta_1(t)$, $\eta_2(t)$, $\beta_0(t)$, $\beta_1(t)$ and $\beta_2(t)$ by virtue of symbolic computation (details ignored), giving rise to three transformations for equation (11), as follows:

The first transformation from equation (11) to equation (13):

$$u^{(1)}(x,t) = c\, e^{-\int h_1(t)\, dt}\, U[\, X(x,t),\, T(t)\,] \times e^{i\left\{[\int \theta(t)\, dt + c_3]x - \int f(t)[\int \theta(t) dt + c_3]^2 dt + \int \sigma(t) dt + c_4\right\}}, \tag{31}$$

with

$$X(x,t) = (ic_1 + c_2)\left\{x - 2\int f(t)\left[\int \theta(t) dt + c_3\right] dt\right\} + c_5, \tag{32}$$

$$T(t) = (i\, c_1 + c_2)^2 \int f(t)\, dt + \delta_1, \tag{33}$$

and $f(t), g_1(t), g_2(t), h_1(t), h_2(x,t)$ satisfying

$$\begin{cases} h_2(x,t) = \theta(t)\, x + \sigma(t), \\ g_1(t) = \pm\, \dfrac{4\, c_1\, c_2}{c^2}\, e^{2\int h_1(t)\, dt} f(t), \\ g_2(t) = \pm\, \dfrac{2\, \left(c_2^2 - c_1^2\right)}{c^2}\, e^{2\int h_1(t) dt} f(t), \end{cases} \tag{34}$$

where δ_1 and c_5 are two arbitrary complex constants, $c \neq 0$, c_1, c_2, c_3 and c_4 are all arbitrary real constants.

The second transformation from equation (11) to equation (13):

$$u^{(2)}(x,t)=\bar{c}e^{-\int[h_1(t)+2f(t)\chi_1(t)]dt}U[X(x,t),T(t)] \times e^{i\left\{\chi_1(t)x^2+\chi_1(t)\chi_2(t)x+\int\left[\sigma(t)-f(t)\chi_1^2(t)\chi_2^2(t)\right]dt+\bar{c}_5\right\}}, \quad (35)$$

with

$$X(x,t) = (i\bar{c}_1+\bar{c}_2)\left[\chi_1(t)x-2\int f(t)\chi_1^2(t)\chi_2(t)dt\right]+\bar{c}_6, \quad (36)$$

$$T(t) = \frac{(\bar{c}_1 - i\,\bar{c}_2)^2\,\chi_1(t)}{4} + \delta_2, \quad (37)$$

and $f(t),\ g_1(t),\ g_2(t),\ h_1(t),\ h_2(x,t)$ satisfying

$$\begin{cases} h_2(x,t) = \theta(t)\,x + \sigma(t), \\ g_1(t) = \pm\dfrac{4\,\bar{c}_1\,\bar{c}_2}{\bar{c}^2}\,\chi_1(t)\,e^{2\int h_1(t)\,dt}f(t), \\ g_2(t) = \pm\dfrac{2\,(\bar{c}_2^2-\bar{c}_1^2)}{\bar{c}^2}\,\chi_1(t)\,e^{2\int h_1(t)\,dt}f(t), \end{cases} \quad (38)$$

where $\chi_1(t)$ and $\chi_2(t)$ are defined as

$$\chi_1(t) = \frac{1}{4\int f(t)\,dt + \bar{c}_3}, \quad \chi_2(t) = \int\frac{\theta(t)}{\chi_1(t)}\,dt + \bar{c}_4, \quad (39)$$

δ_2 and $\bar{c}_6$ are two arbitrary complex constants, $\bar{c} \neq 0$, $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$, $\bar{c}_4$ and $\bar{c}_5$ are all arbitrary real constants.

The third transformation from equation (11) to equation (13):

$$u^{(3)}(x,t) = \tilde{c}\,e^{-\int[\,h_1(t)+2\,f(t)\,\beta_2(t)\,]\,dt}\,U[\,X(x,t),\,T(t)\,] \times e^{i\left\{\beta_2(t)\,x^2+\chi_3(t)\,\chi_4(t)\,x+\int\left[\,\sigma(t)-f(t)\,\chi_3^2(t)\,\chi_4^2(t)\,\right]dt+\tilde{c}_5\right\}}, \quad (40)$$

with

$$X(x,t) = (i\,\tilde{c}_1 + \tilde{c}_2)\left[\chi_3(t)\,x - 2\int f(t)\,\chi_3^2(t)\,\chi_4(t)\,dt\right] + \tilde{c}_6, \quad (41)$$

$$T(t) = (i\,\tilde{c}_1 + \tilde{c}_2)^2\int f(t)\,\chi_3^2(t)\,dt + \delta_3, \quad (42)$$

and $f(t),\ g_1(t),\ g_2(t),\ h_1(t), h_2(x,t)$ satisfying

$$\begin{cases} h_2(x,t) = \vartheta(t)\,x^2 + \theta(t)\,x + \sigma(t), \\ g_1(t) = \pm\dfrac{4\,\tilde{c}_1\,\tilde{c}_2}{\tilde{c}^2}\,\chi_3(t)\,e^{2\int h_1(t)\,dt}f(t), \\ g_2(t) = \pm\dfrac{2\,(\tilde{c}_2^2-\tilde{c}_1^2)}{\tilde{c}^2}\,\chi_3(t)\,e^{2\int h_1(t)\,dt}f(t), \end{cases} \quad (43)$$

where $\chi_3(t)$ and $\chi_4(t)$ are defined as

$$\chi_3(t) = e^{-4\int f(t)\,\beta_2(t)\,dt}, \quad \chi_4(t) = \int\frac{\theta(t)}{\chi_3(t)}\,dt + \tilde{c}_4, \quad (44)$$

δ_3 and $\tilde{c}_6$ are two arbitrary complex constants, $\tilde{c} \neq 0$, $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{c}_3$, $\tilde{c}_4$ and $\tilde{c}_5$ are all arbitrary real constants, and $\beta_2(t)$ satisfies

$$\beta_2'(t) + 4\,f(t)\,\beta_2^2(t) - \vartheta(t) = 0 \quad \text{with} \quad \vartheta(t) \neq 0. \quad (45)$$

Here, it is a formidable task to give the general solution for $\beta_2(t)$ because equation (45) is a Ricatti equation with two arbitrary coefficient functions $f(t)$ and $\vartheta(t)$. However, equation (45) indeed includes the most general cases of transforming equation (11) into equation (13) when $h_{2,xx} \neq 0$, for example, equations (48) and (49) as below.

2.2 Comparisons with other existing results

For equation (11), conditions (34), (38) and (43) constitute its most general cases to be transformable into the standard NLS equation. Compared with many previously-published papers, the above-obtained transformations and relevant constraint conditions are found to be more broader than other existing results:

1. When $g_1(t) = 0$ and $h_2(x,t) = 0$, via the first two transformations in the above subsection, we are actually able to get four transformations for equation (10) to be mapped onto the standard NLS equation, i.e., transformation (31) with $c_1 = 0$ and $c_2 \neq 0$, transformation (35) with $\bar{c}_1 = 0$ and $\bar{c}_2 \neq 0$, transformation (31) with $c_1 \neq 0$ and $c_2 = 0$, and transformation (35) with $\bar{c}_1 \neq 0$ and $\bar{c}_2 = 0$, where the first two cases have been given in reference [12] (i.e., transformations A and B there), while the last two cases have not been reported as far as we know.
2. As stated in reference [29], the following variable-coefficient NLS equation which is a special case of equation (11),

$$i\,u_t + p(t)\,u_{xx} + q(t)\,|u|^2\,u = 0, \quad (46)$$

can be transformed into the standard one when $p(t)$ and $q(t)$ satisfy the Painlevé-integrable condition of equation (46),

$$p(t) = q(t)\left[a_1\int p(t)\,dt + a_2\right], \quad (47)$$

with a_1 and a_2 as two arbitrary real constants, where $a_1 = 0$ and $a_1 \neq 0$ correspond to the special cases of conditions (34) and (38), respectively.
3. References [30,33] have given another two special variable-coefficient NLS equations which are both transformable into the standard one, as follows:

$$i\,u_t + u_{xx} - 2\,|u|^2\,u = a(x,t)\,u + i\,\beta_3(t)\,u, \quad (48)$$

and

$$i\,u_t + u_{xx} + \tilde{g}_1\,T_0\,\mathrm{sech}(\sqrt{K_0}\,t)\,|u|^2\,u + \frac{K_0}{4}\,x^2\,u = 0, \quad (49)$$

$$\begin{pmatrix}\phi_1^{(2)}\\ \phi_2^{(2)}\end{pmatrix}_t = \begin{pmatrix}(\bar{c}_1 - i\,\bar{c}_2)\,f(t)\,\chi_1^2(t)\,A_2 & -(\bar{c}_1 - i\,\bar{c}_2)\,f(t)\,\chi_1(t)\,\rho_2\,B_2\\ \pm(\bar{c}_1 - i\,\bar{c}_2)\,f(t)\,\chi_1(t)\,\rho_2^*\,C_2 & -(\bar{c}_1 - i\,\bar{c}_2)\,f(t)\,\chi_1^2(t)\,A_2\end{pmatrix}\begin{pmatrix}\phi_1^{(2)}\\ \phi_2^{(2)}\end{pmatrix}, \tag{60}$$

$$\rho_2 = \frac{1}{\bar{c}}\,e^{\int[\,h_1(t)+2\,f(t)\,\chi_1(t)\,]\,dt - i\left\{\chi_1(t)\,x^2+\chi_1(t)\,\chi_2(t)\,x+\int\left[\,\sigma(t)-f(t)\,\chi_1^2(t)\,\chi_2^2(t)\right]dt+\bar{c}_5\right\}}, \tag{61}$$

where $a(x,t)$ is defined as

$$a(x,t) = \left[\frac{1}{2}\,\beta_3' - \beta_3^2(t)\right]x^2 + \alpha_1(t)\,x + \alpha_2(t), \tag{50}$$

$\beta_3(t)$, $\alpha_1(t)$ and $\alpha_2(t)$ are all arbitrary analytical functions, K_0, T_0 and $\tilde{g}_1$ are all real constants. Through symbolic substitution, it can be easily proved that the coefficient functions in equations (48) and (49) satisfy conditions (43).

For other variable-coefficient NLS-typed equations which can be transformed into equation (13), we refer the readers to references [4,7,8,10,19,23,31,32] and references therein.

3 Lax pairs for equation (11) via symbolic computation

Each of the three transformations obtained in Section 2 maps a subclass of the solutions of equation (11) onto equation (13) whose initial value problem is solvable through the method of inverse scattering [1]. Since the properties and solutions of equation (13) have been studied in great detail, as seen, e.g., in references [1,2] and references therein, the investigations for many variable-coefficient NLS-typed equations in which the coefficient functions satisfy conditions (34), (38) or (43) can be based on equation (13).

When $g_1(t) = 0$, using transformations (31), (35) and (40), we can easily derive the linear scattering problems (i.e., Lax pairs) for equation (11) respectively under conditions (34), (38) and (43). It is well-known that equation (13) is associated with the following linear scattering problem [1,2], expressed as

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_X = \begin{pmatrix}-i\,\lambda & U\\ \mp U^* & i\,\lambda\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{51}$$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_T = \begin{pmatrix}-2\,i\,\lambda^2 \pm i\,U\,U^* & 2\,\lambda\,U + i\,U_X\\ \mp 2\,\lambda\,U^* \pm i\,U_X^* & 2\,i\,\lambda^2 \mp i\,U\,U^*\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{52}$$

where ϕ_1 and ϕ_2 are two components of the vector eigenfunction, λ is the spectral parameter which is a complex constant, and the star superscript denotes the complex conjugate. Combining equations (51) and (52) with transformations (31), (35) and (40), after tedious substitution, derivation, simplification and other manipulations on the computerized symbolic computation system (details ignored), we have three linear scattering problems for equation (11):

The first Lax pair corresponding to conditions (34) with $c_1 = 0$ or $c_2 = 0$:

$$\begin{pmatrix}\phi_1^{(1)}\\ \phi_2^{(1)}\end{pmatrix}_x = \begin{pmatrix}\lambda\,(c_1 - i\,c_2) & (i\,c_1 + c_2)\,\rho_1\,u\\ \mp(i\,c_1 + c_2)\,\rho_1^*\,u^* & -\lambda\,(c_1 - i\,c_2)\end{pmatrix}\begin{pmatrix}\phi_1^{(1)}\\ \phi_2^{(1)}\end{pmatrix}, \tag{53}$$

$$\begin{pmatrix}\phi_1^{(1)}\\ \phi_2^{(1)}\end{pmatrix}_t = \begin{pmatrix}(c_1 - ic_2)\,f(t)A_1 & -(c_1 - ic_2)\,f(t)\rho_1 B_1\\ \pm(c_1 - ic_2)\,f(t)\rho_1^* C_1 & -(c_1 - ic_2)\,f(t)A_1\end{pmatrix}\begin{pmatrix}\phi_1^{(1)}\\ \phi_2^{(1)}\end{pmatrix}, \tag{54}$$

with

$$\rho_1 = \frac{1}{c}\,e^{\int h_1(t)\,dt - i\left\{[\int\theta(t)\,dt+c_3]\,x-\int f(t)\,[\int\theta(t)\,dt+c_3]^2\,dt+\int\sigma(t)\,dt+c_4\right\}}, \tag{55}$$

$$A_1 = \mp\frac{(i\,c_1 + c_2)\;u\,u^*}{c^2}\,e^{2\int h_1(t)\,dt} - 2\,\lambda\left[\int\theta(t)\,dt - i\,\lambda\,c_1 - \lambda\,c_2 + c_3\right], \tag{56}$$

$$B_1 = \left[i\int\theta(t)\,dt + 2\,\lambda\,c_1 - 2\,i\,\lambda\,c_2 + i\,c_3\right]u + u_x\,, \tag{57}$$

$$C_1 = \left[i\int\theta(t)\,dt + 2\,\lambda\,c_1 - 2\,i\,\lambda\,c_2 + i\,c_3\right]u^* - u_x^*\,. \tag{58}$$

The second Lax pair corresponding to conditions (38) with $\bar{c}_1 = 0$ or $\bar{c}_2 = 0$:

$$\begin{pmatrix}\phi_1^{(2)}\\ \phi_2^{(2)}\end{pmatrix}_x = \begin{pmatrix}\lambda\,(\bar{c}_1 - i\,\bar{c}_2)\,\chi_1(t) & (i\,\bar{c}_1 + \bar{c}_2)\,\chi_1(t)\,\rho_2\,u\\ \mp(i\,\bar{c}_1 + \bar{c}_2)\,\chi_1(t)\,\rho_2^*\,u^* & -\lambda\,(\bar{c}_1 - i\,\bar{c}_2)\,\chi_1(t)\end{pmatrix}\begin{pmatrix}\phi_1^{(2)}\\ \phi_2^{(2)}\end{pmatrix}, \tag{59}$$

see equation (60) above

with

see equation (61) above

$$\begin{pmatrix}\phi_1^{(3)}\\ \phi_2^{(3)}\end{pmatrix}_t = \begin{pmatrix}(\tilde{c}_1 - i\,\tilde{c}_2)\,f(t)\,\chi_3^2(t)\,A_3 & -(\tilde{c}_1 - i\,\tilde{c}_2)\,f(t)\,\chi_3^2(t)\,\rho_3\,B_3\\ \pm(\tilde{c}_1 - i\,\tilde{c}_2)\,f(t)\,\chi_3^2(t)\,\rho_3^*\,C_3 & -(\tilde{c}_1 - i\,\tilde{c}_2)\,f(t)\,\chi_3^2(t)\,A_3\end{pmatrix}\begin{pmatrix}\phi_1^{(3)}\\ \phi_2^{(3)}\end{pmatrix}, \tag{66}$$

$$\rho_3 = \frac{1}{\tilde{c}}\,e^{\int[h_1(t)+2f(t)\beta_2(t)]\,dt - i\left\{\beta_2(t)\,x^2+\chi_3(t)\,\chi_4(t)\,x+\int\left[\sigma(t)-f(t)\,\chi_3^2(t)\,\chi_4^2(t)\right]dt+\tilde{c}_5\right\}}, \tag{67}$$

$$A_2 = \mp\frac{(i\,\bar{c}_1 + \bar{c}_2)\,u\,u^*}{\bar{c}^2}e^{2\int[h_1(t)+2f(t)\chi_1(t)]dt} - 2\lambda\,[2x + \chi_2(t) - i\lambda\bar{c}_1 - \lambda\bar{c}_2], \tag{62}$$

$$B_2 = i\,[2x + \chi_2(t) - 2i\lambda\bar{c}_1 - 2\lambda\bar{c}_2]\,\chi_1(t)u + u_x, \tag{63}$$

$$C_2 = i\,[2x + \chi_2(t) - 2i\lambda\bar{c}_1 - 2\lambda\bar{c}_2]\,\chi_1(t)u^* - u_x^*, \tag{64}$$

where $\chi_1(t)$ and $\chi_2(t)$ have been defined in equations (39).

The third Lax pair corresponding to conditions (43) with $\tilde{c}_1 = 0$ or $\tilde{c}_2 = 0$:

$$\begin{pmatrix}\phi_1^{(3)}\\ \phi_2^{(3)}\end{pmatrix}_x = \begin{pmatrix}\lambda\,(\tilde{c}_1 - i\,\tilde{c}_2)\,\chi_3(t) & (i\,\tilde{c}_1 + \tilde{c}_2)\,\chi_3(t)\,\rho_3\,u\\ \mp(i\,\tilde{c}_1 + \tilde{c}_2)\,\chi_3(t)\,\rho_3^*\,u^* & -\lambda\,(\tilde{c}_1 - i\,\tilde{c}_2)\,\chi_3(t)\end{pmatrix}\begin{pmatrix}\phi_1^{(3)}\\ \phi_2^{(3)}\end{pmatrix}, \tag{65}$$

see equation (66) above

with

see equation (67) above

$$A_3 = \mp\frac{(i\tilde{c}_1 + \tilde{c}_2)uu^*}{\tilde{c}^2}e^{2\int[h_1(t)+2f(t)\beta_2(t)]dt} - 2\lambda\left[\frac{2\beta_2(t)}{\chi_3(t)}x + \chi_4(t) - i\lambda\tilde{c}_1 - \lambda\tilde{c}_2\right], \tag{68}$$

$$B_3 = [i\chi_4(t) + 2\lambda\tilde{c}_1 - 2i\lambda\tilde{c}_2]\,u + \frac{2ix\beta_2(t)u + u_x}{\chi_3(t)}, \tag{69}$$

$$C_3 = [i\chi_4(t) + 2\lambda\tilde{c}_1 - 2i\lambda\tilde{c}_2]\,u^* + \frac{2ix\beta_2(t)u^* - u_x^*}{\chi_3(t)}, \tag{70}$$

where $\chi_3(t)$ and $\chi_4(t)$ have been defined in equations (44), and $\beta_2(t)$ satisfies equation (45).

It is easy to verify that the compatibility conditions $\phi_{1,xt}^{(j)} = \phi_{1,tx}^{(j)}$ and $\phi_{2,xt}^{(j)} = \phi_{2,tx}^{(j)}$ ($j = 1, 2$ and 3) lead to equation (11) with the respective constraint conditions, which suggests that equation (11) with $g_1(t) = 0$ is integrable provided that conditions (34), (38) or (43) are satisfied. Besides, we recall that the three Lax pairs presented above are all the isospectral scattering problems.

When $g_1(t) \neq 0$, it is unsuccessful in attempting to employ transformations (31), (35) and (40) to obtain the Lax pairs for equation (11) under conditions (34), (38) or (43) from equations (51) and (52), because the compatibility conditions yield redundant equations except equation (11). We think that the failure is caused by the transformations of dependent variables from the real-valued space to complex-valued one due to the existence of the term $i\,g_1(t)\,|u|^2\,u$ in equation (11). More detailed explanations need to be further explored.

4 Applications and discussions

The coefficient functions in equation (11), although hard to be dealt with, play a practical role in the description for various nonlinear wave phenomena in diverse inhomogeneous media. The construction of three transformations from equation (11) to the standard NLS equation makes it possible to study many variable-coefficient NLS-typed models under conditions (34), (38) or (43). In this section, to further show the physical relevance of transformations (31), (35) and (40), we will apply the three transformations to some example models aforementioned and discuss relevant physical mechanism.

4.1 Stable curve soliton excitations in arbitrary linearly inhomogeneous plasmas

Generally, the electromagnetic wave propagation in plasmas is strongly affected by both the nonlinearity and nonuniformity of the medium [4]. In the presence of a linear, time-dependent density profile, the propagation of an electron plasma wave packet with very large wavelength and finite amplitude is described by a special form of equation (2) [4,5],

$$i\,q_t + q_{xx} + \left[2\,|q|^2 - \bar{\alpha}(t)\,x\right]q = 0, \tag{71}$$

where $\bar{\alpha}(t)$ is an arbitrary analytical function of t. In this case, we can utilize transformation (31) to derive the following bright one-soliton-like solution:

$$q = \bar{A}_1\,\mathrm{sech}\left[\bar{A}_1\,\xi_1(x,t)\right]\,e^{i\,\omega_1(x,t)+i\,(\bar{A}_1^2-k_1^2)\,t}, \tag{72}$$

$$\xi_1(x,t) = x - 2\,k_1\,t + 2\int\left(\int\bar{\alpha}(t)\,dt\right)dt, \tag{73}$$

$$\omega_1(x,t) = \left[k_1 - \int\bar{\alpha}(t)\,dt\right]x + 2\,k_1\int\left(\int\bar{\alpha}(t)\,dt\right)dt - \int\left(\int\bar{\alpha}(t)\,dt\right)^2 dt, \tag{74}$$

where $\bar{A}_1$ and k_1 are two nonzero real constants. Obviously, it can be seen from solution (72) that the soliton amplitude $|\bar{A}_1|$ is always kept invariant, while the soliton velocity and acceleration are respectively as follows:

$$v = 2\,k_1 - 2\int\bar{\alpha}(t)\,dt \quad\text{and}\quad \dot{v} = -2\,\bar{\alpha}(t), \tag{75}$$

which suggest that the linearly inhomogeneous coefficient $\bar{\alpha}(t)$ leads to the accelerated propagation of nonlinear wave packet. Figure 1, with $\bar{\alpha}(t)$ chosen as a constant,

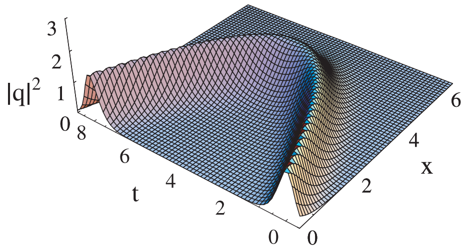

Fig. 1. Intensity profile of a parabolic soliton via solution (72) for $\bar{\alpha}(t) = 0.25$ with the acceleration $\dot{v} = -0.5$. The related parameters are chosen as $\bar{A}_1 = 1.24$ and $k_1 = 1$.

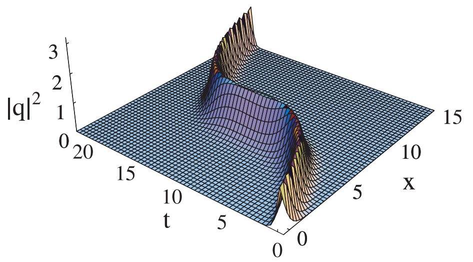

Fig. 2. Intensity profile of a cubic soliton via solution (72) for $\bar{\alpha}(t) = -0.025\,t + 0.225$ with the acceleration $\dot{v} = 0.05\,t - 0.45$. The choice of other parameters follows Figure 1.

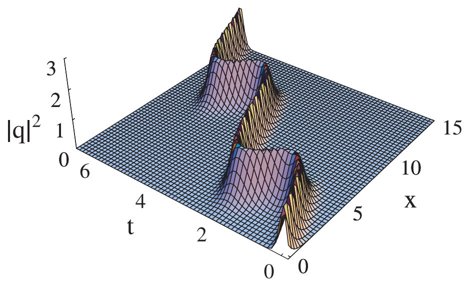

Fig. 3. Intensity profile of a periodically oscillating soliton via solution (72) for $\bar{\alpha}(t) = 4\sin(2\,t)$ with the acceleration $\dot{v} = -8\sin(2\,t)$. The choice of other parameters follows Figure 1.

shows that the soliton is uniformly accelerated with the same acceleration in a linear, time-independent density profile. When $\bar{\alpha}(t) \neq$ constant, Figures 2 and 3 display a couple of nonuniformly accelerated soliton structures, corresponding to two different linear time-dependent inhomogeneities. It is also noted that $\int_{-\infty}^{+\infty} |q|^2\,dx$ is a conserved quantity of equation (71). Therefore, we know that the linear inhomogeneity $\bar{\alpha}(t)\,x$ does not affect the integrability of equation (71), and that the stable propagation of the soliton with unchangeable amplitude, width and shape (but the velocity is exceptional) is supported in an arbitrary linearly inhomogeneous plasma.

4.2 Stationary localized envelope pulses with the linear fiber loss/gain and frequency chirping

In the case of an interplay between the linear fiber loss/gain and initial chirping, the pulse propagation in such a fiber is characterized by a special form of equation (3) [34]

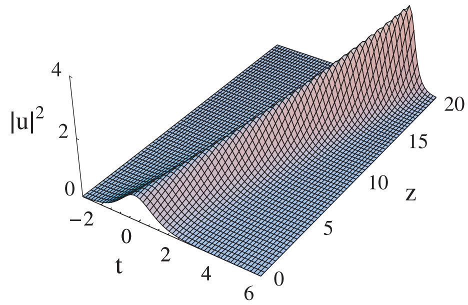

Fig. 4. A propagating envelope pulse with the linear fiber gain and frequency chirp. The related parameters are chosen as $\bar{A}_2 = 0.95$, $k_2 = 0.1$ and $\bar{\beta} = -0.015$.

$$i\,u_z + u_{tt} + 2\,|u|^2\,u + i\,\gamma\,u + \bar{\beta}^2\,t^2\,u = 0, \qquad (76)$$

where $\bar{\beta}^2\,t^2\,u$ is the quadratic phase chirp term, $\gamma > 0$ (or $\gamma < 0$) accounts for the linear loss (or gain) rate. Via transformation (40), equation (76) can be reduced to

$$i\,U_Z + U_{TT} + 2\,|U|^2\,U = i\,e^{4\,\bar{\beta}\,z}\left(\bar{\beta} - \gamma\right)U, \qquad (77)$$

with $Z = (1 - e^{-4\,\bar{\beta}\,z})/4\,\bar{\beta}$, $T = e^{-2\,\bar{\beta}\,z}\,t$ and $u(t,z) = e^{\frac{i}{2}\,\bar{\beta}\,t^2 - 2\,\bar{\beta}\,z}\,U(T,Z)$. It is obvious that equation (77) becomes the standard NLS equation when $\gamma = \bar{\beta}$ which means the exact balance between the linear fiber loss/gain and pulse chirp terms. Similarly, we can also obtain the bright one-soliton-like solution for equation (76) with $\gamma = \bar{\beta}$ as below,

$$u = \bar{A}_2 e^{-2\bar{\beta}z}\mathrm{sech}\left[\bar{A}_2 e^{-2\bar{\beta}z}\xi_2(t,z)\right]e^{i\omega_2(t,z) + i\frac{1}{2}\bar{\beta}t^2}, \qquad (78)$$

$$\xi_2(t,z) = t - \frac{k_2}{2\,\bar{\beta}}(e^{2\,\bar{\beta}\,z} - e^{-2\,\bar{\beta}\,z}), \qquad (79)$$

$$\omega_2(t,z) = k_2\,e^{-2\,\bar{\beta}\,z}\,t + \frac{\bar{A}_2^2 - k_2^2}{4\,\bar{\beta}}(1 - e^{-4\,\bar{\beta}\,z}), \qquad (80)$$

with the amplitude $|\bar{A}_2|\,e^{-2\,\bar{\beta}\,z}$ and the width $|\bar{A}_2|^{-1}\,e^{2\,\bar{\beta}\,z}$, where $\bar{A}_2$ and k_2 are two nonzero real constants.

For different signs of $\bar{\beta}$, there are two types of pulse propagating modes. In Figure 4, it can be observed that the pulse amplitude monotonously grows owing to the fiber gain while the pulse width gets narrow as a result of the frequency chirping both in an exponential way during the propagation. Figure 5 illustrates the effect of a pulse with amplitude attenuating and width broadening as it progresses along the length of the fiber. Our analysis reveals that the envelope pulses described by equation (76) with $\gamma = \bar{\beta}$ are stationary localized objects which in practice are more attractive in realistic optical communication systems [18]. The localized stability of such pulses could be explained from the nonconservation of its physical quantities. Through symbolic calculations, the first conservation law of equation (76) with $\gamma = \bar{\beta}$ can be given as

$$i\,\frac{\partial\left(e^{2\,\bar{\beta}\,z}\,|u|^2\right)}{\partial z} + \frac{\partial\left[e^{2\,\bar{\beta}\,z}\left(u_t\,u^* - u_t^*\,u\right)\right]}{\partial t} = 0, \qquad (81)$$

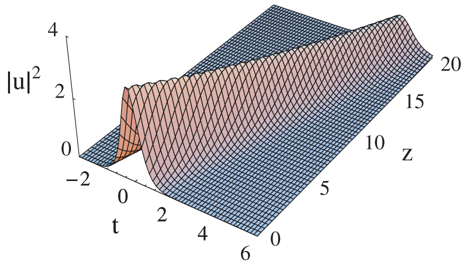

Fig. 5. A propagating envelope pulse with the linear fiber loss and frequency chirp. The related parameters are chosen as $\bar{A}_2 = 1.75$, $k_2 = 0.1$ and $\bar{\beta} = 0.012$.

where $e^{2\bar{\beta} z}|u|^2$ and $e^{2\bar{\beta} z}(u_t u^* - u_t^* u)$, respectively, correspond to the conserved density and flux with a modification by multiplication of $e^{2\bar{\beta} z}$ to counteract the attenuation/growth caused by the linear fiber loss/gain. Using the vanishing boundary condition for solution (78), we can compute out the corresponding conserved quantity,

$$\int_{-\infty}^{+\infty} e^{2\bar{\beta} z}|u|^2\, dt = 2\,|\bar{A}_2|, \tag{82}$$

which indicates that the quantity $\int_{-\infty}^{+\infty}|u|^2\,dt$ will exponentially decay/grow as the rate $e^{-2\bar{\beta} z}$.

In real optical fibers, the chirping parameter may not be equal to the square of the linear loss/gain rate, i.e., $\gamma \neq \bar{\beta}$. However, we notice a fact that if $\bar{\beta} < 0$ the right hand of equation (77) will near zero as the normalized distance z increases, which implies that solution (78) with $\bar{\beta} < 0$ can be used as an approximate solution for equation (76) when $\gamma \neq \bar{\beta}$.

4.3 Optical pulses with variable dispersion, Kerr nonlinearity and loss/gain

Considering the variability of group velocity dispersion, self-phase modulation, linear and nonlinear loss/gain, the pulse dynamics in such a monomode optical fiber is governed by equation (4) with distributed coefficients [11], for which there are two cases to be transformed into equation (13), as follows:

$$\Pi(z) = \pm\frac{2\,b_1\,b_2}{b^2}\,e^{2\int\Gamma(z)\,dz}D(z), \tag{83}$$

$$R(z) = \pm\frac{(b_2^2 - b_1^2)}{b^2}\,e^{2\int\Gamma(z)\,dz}D(z), \tag{84}$$

and

$$\Pi(z) = \pm\frac{2\,b_1\,b_2}{b^2}\,e^{2\int\Gamma(z)\,dz}D(z)\,\chi(z), \tag{85}$$

$$R(z) = \pm\frac{(b_2^2 - b_1^2)}{b^2}\,e^{2\int\Gamma(z)\,dz}D(z)\,\chi(z), \tag{86}$$

where $\chi(z)$ is defined as

$$\chi(z) = \frac{1}{2\int D(z)\,dz + b_3}, \tag{87}$$

and $b \neq 0$, b_1, b_2 and b_3 are all arbitrary real constants.

If $b_1 = 0$, it is noted that conditions (84) and (86) actually constitute the integrable cases for equation (4) with $\Pi(z) = 0$ to pass the Painlevé test [35], which suggests that the generation of optical solitons is still based on the balance among the coefficient functions in addition to the exact balance between the group velocity dispersion and self-phase modulation. On the other hand, we notice that there exists only one constraint equation among $D(z)$, $R(z)$ and $\Gamma(z)$, i.e., condition (84) or (86). Accordingly, with different choices of the coefficient functions, a rich class of soliton-like solutions can be obtained for describing various fiber systems such as periodically-varying management system [21], exponentially distributed control system [15] and dispersion decreasing fiber system with nonlinear barriers [17]. Under condition (84) or (86), the first conservation law of equation (4) with $\Pi(z) = 0$ can be derived as

$$i\,\frac{\partial\left(e^{2\int\Gamma(z)\,dz}|u|^2\right)}{\partial z} + \frac{\partial\left[\frac{1}{2}D(z)\left(u_t u^* - u_t^* u\right)\right]}{\partial t} = 0, \tag{88}$$

where $e^{2\int\Gamma(z)\,dz}|u|^2$ and $\frac{1}{2}D(z)(u_t u^* - u_t^* u)$, respectively, correspond to the conserved density and flux. Here, we can see that for equation (4) under condition (84) or (86) with $b_1 = 0$, the soliton-like solution is stable without the linear loss/gain term while with inclusion of $\Gamma(z)$ the soliton-like solution is a localized object. For nonintegrable conditions, we consider the following general case:

$$i\,u_z + \frac{D(z)}{2}\,u_{tt} + e^{2\int\Gamma(z)\,dz}D(z)\,|u|^2\,u + i\,r(z)\,u = 0, \tag{89}$$

which can be transformed into

$$i\,U_Z + U_{TT} + 2\,|U|^2\,U = i\,\frac{2[\,\Gamma(z) - r(z)]}{D(z)}\,U, \tag{90}$$

with $u(t,z) = e^{i\,[t-\frac{1}{2}\int D(z)\,dz] - \int\Gamma(z)\,dz}\,U(T,Z)$, $T = t - \int D(z)\,dz$ and $Z = \frac{1}{2}\int D(z)\,dz$. When $\left|[\,\Gamma(z) - r(z)]/D(z)\right| \ll 1$, the solutions of equation (89) can be approximated by the solutions of equation (4) with $R(z) = e^{2\int\Gamma(z)\,dz}D(z)$ and $\Pi(z) = 0$.

From the integrable viewpoint, recent applications and developments of equation (4) in the absence of the nonlinear loss/gain term include the following:

(a) For condition (86) with $D(z) = R(z) = 1$ and $b_1 = 0$, reference [13] has found a growing, chirped, soliton-like solution without phase modulation which is self-similar and suggests the possibility of clean and efficient nonlinear compression of chirped solitary waves with appropriate tailoring of the gain as a function of distance.

(b) In reference [21], a systematic way has been developed to find an infinite number of novel stable bright and dark "soliton islands" in a "sea of solitary waves" of equation (4) with $\Pi(z) = 0$ under certain conditions (i.e., conditions (3) and (5) there) which correspond to two special cases of condition (86). Some special soliton-like solutions have also been constructed

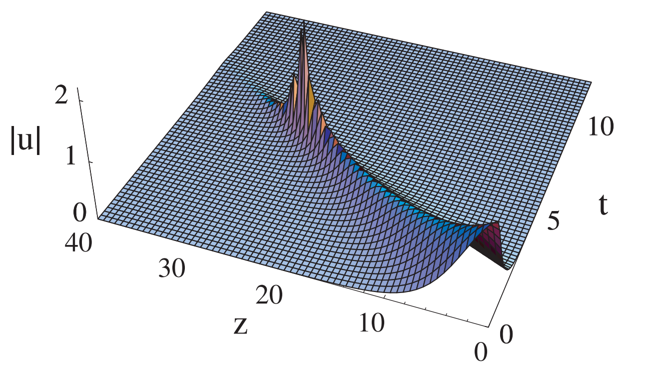

Fig. 6. Amplitude profile of an unstable optical pulse with the related parameters and functions chosen as $\bar{A}_3 = 1.75$, $k_3 = 1$, $b_1 = 0.15$, $b_2 = 0.75$, $D(z) = 0.15$ and $\Gamma(z) = 0$.

for different soliton management conditions including the soliton dispersion management, soliton amplification management, soliton pulse width management, and combined nonlinear and dispersion soliton management.

(c) Under condition (86), reference [15] has employed the transformation from equation (4) with $\Pi(z) = 0$ to the known NLS equation and obtained the bright soliton on a continuous wave background for describing the modulation instability in inhomogeneous fibers.

(d) When condition (84) is satisfied, reference [17] has investigated the nonlinear tunneling of optical solitons through both dispersion and nonlinear barriers described by equation (4) with $\Pi(z) = 0$, and presented a cascade compression system in a dispersion decreasing fiber with nonlinear barriers on an exponential background.

If $\Pi(z) \neq 0$, equation (4) is a nonintegrable model which has not possess soliton solutions [16]. Under conditions (83)–(84) or (85)–(86), the solutions of equation (4) can be directly obtained by virtue of the known solutions of equation (13). But in such a way, stable solutions could be mapped onto the unstable ones because of the extension of our transformations to the complex domain. For example, from the bright one-soliton solution of equation (13), we can obtain the following solution

$$u = \bar{A}_3\, e^{-\int \Gamma(z)\, dz}\, \operatorname{sech}\left[\bar{A}_3\, (i\, b_1 + b_2)\; \xi_3(t,z)\right] \times e^{i\, \omega_3(t,z)}, \tag{91}$$

$$\xi_3(t,z) = t - k_3\, (i\, b_1 + b_2) \int D(z)\, dz, \tag{92}$$

$$\omega_3(t,z) = k_3\, (i\, b_1 + b_2)\, t + \frac{1}{2}(i\, b_1 + b_2)^2\, (\bar{A}_3^2 - k_3^2) \times \int D(z)\, dz, \tag{93}$$

the modulus of which is unstable along the propagation due to the existence of a singularity (see Fig. 6), where $\bar{A}_3$ and k_3 are two nonzero real constants. From conditions (83) and (85), we know that the nonlinear loss/gain coefficient is in proportion to the Kerr nonlinearity coefficient. In the case that $\Pi(z)$ is enough weak compared to $R(z)$, we choose a very small value for b_1 and present Figures 7 and 8, from which it can be seen that the shape of the optical pulse keeps invariant for a considerable length along the fiber and the location where the singularity takes place is transferred away.

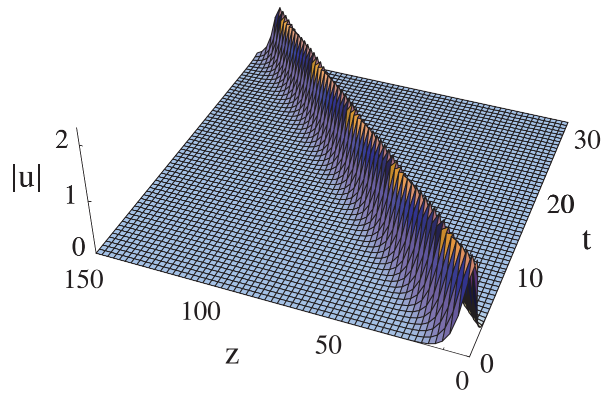

Fig. 7. Stable propagation of an optical pulse in a finite length of the fiber with weak nonlinear loss effect. The parameters and functions are the same as those in Figure 6 except that $b_1 = 0.008$.

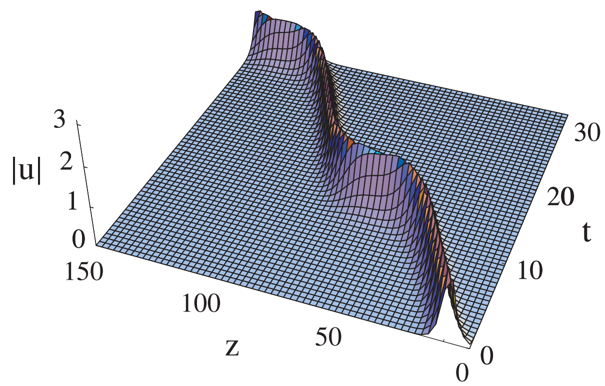

Fig. 8. Stable propagation of an optical pulse in a finite length of the fiber with weak nonlinear loss effect and periodically-varying distributed coefficients. The parameters and functions are the same as those in Figure 7 except that $D(z) = 0.14\sin(0.08\, z) + 0.15$.

5 Conclusions

Taking into account the inhomogeneities of media and nonuniformities of boundaries, many variable-coefficient NLS-typed models have been proposed in various physical contexts like the envelope soliton excitation in inhomogeneous plasmas, pulse dynamics in a real optical-fiber medium or an averaged, dispersion-managed optical-fibre system, mechanism of pulse compression in inhomogeneous optical-transmission systems, arterial mechanics of blood flow, dynamics of Bose-Einstein condensates in a small harmonic trap, etc. In this paper, based on the work of Tian et al. in reference [12], we have investigated all the cases for a more generalized variable-coefficient NLS equation, i.e., equation (11), to be mapped onto the standard one. Performing symbolic computation, we have constructed three transformations from equation (11) to equation (13) with the respective constraint conditions which recover many previously-published results. Through the three transformations, some integrable properties of equation (13) can be directly introduced into equation (11) with $g_1(t) = 0$ if the relevant constraint conditions hold, e.g., the Lax pairs presented in Section 3. We have applied our results to three currently-important example models and detailed the following: (1) stable curve soliton excitations in arbitrary linearly inhomogeneous plasmas; (2) stationary localized envelope pulses with the linear fiber loss/gain and frequency chirping; and (3) propagation of optical pulses with variable dispersion, Kerr nonlinearity and loss/gain. The physical meaning of constraint conditions among the coefficient functions and the stability of soliton-like solutions have also been discussed.

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